Basic ZX-calculus for students and professionals

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Abstract

These are the lecture notes of guest lectures for Artur Ekert's course Introduction to Quantum Information at the Mathematical Institute of Oxford University, Hilary Term 2023. Some basic familiarity with Dirac notation is assumed.

For the readers of Quantum in Pictures who have some basic quantum background, these notes also constitute the shortest path to an explanation of how what they learn in QIP relates to the traditional quantum formalism.

1 Introduction

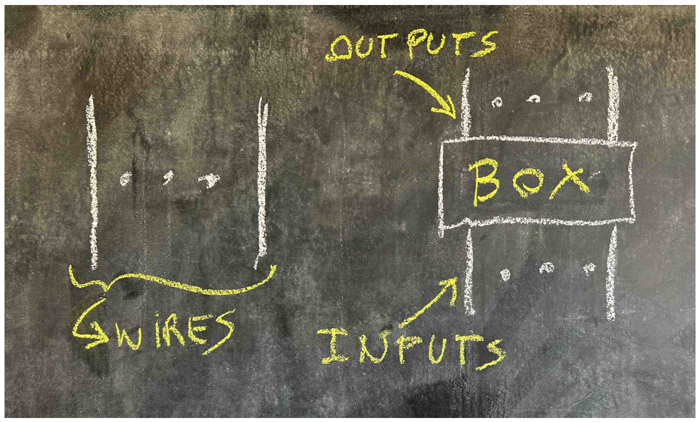
ZX-calculus [7, 8, 9] has already been around as long as 2007, but it only has become widely used in the past couple of years, and especially with the emergence of quantum industry, given the new scientific and technological challenges this poses [12]. Areas where it has become prominent include compilation [17, 27], circuit optimisation [20, 28], error-correction [18, 22], quantum natural language processing [6], QML [35, 34], and also many issues surrounding photonic quantum computing [19, 29].

Another reason for this more recent rise to prominence is the availability of the book Picturing Quantum Processes [14], which presents ZX-calculus without any reference to category theory, which before was an obstacle to many. Even more so, since very recently there also is the book Quantum in Pictures [11] which has no mathematical prerequisites whatsoever. Hence, this book also establishes ZX-calculus as an educational tool that can importantly contribute to making quantum more inclusive, while at the same time providing an entirely new perspective on it.

2 Process theories as wires and boxes

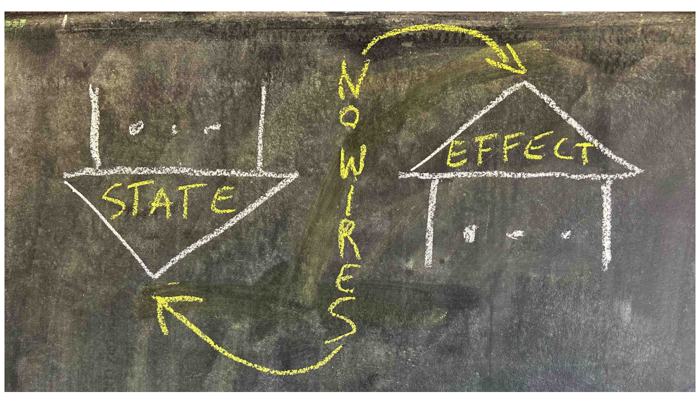
ZX-calculus is an instance of what in [14] we call a process theory, that is, a theory in which processes are first class citizens. Historically, going 2500 years back there was a debate between pre-Socratics Heraclitus and Parmenides, where the former advocated that all there is are processes, with a world in constant flux, while Parmenides advocated that at any time the world is a static instance. Plato, Aristotle, Kepler, Newton, etc. all followed the Parmenides line of thinking, having a kinematic description of the world before dynamic evolution is discussed. Whitehead was the 1st to bring Heraclitus' process ideas back. With the birth of category theory, the appropriate mathematics to do so is now also around, and can equivalently be described as pictures [26].

The language of process theories consists of wires and boxes:

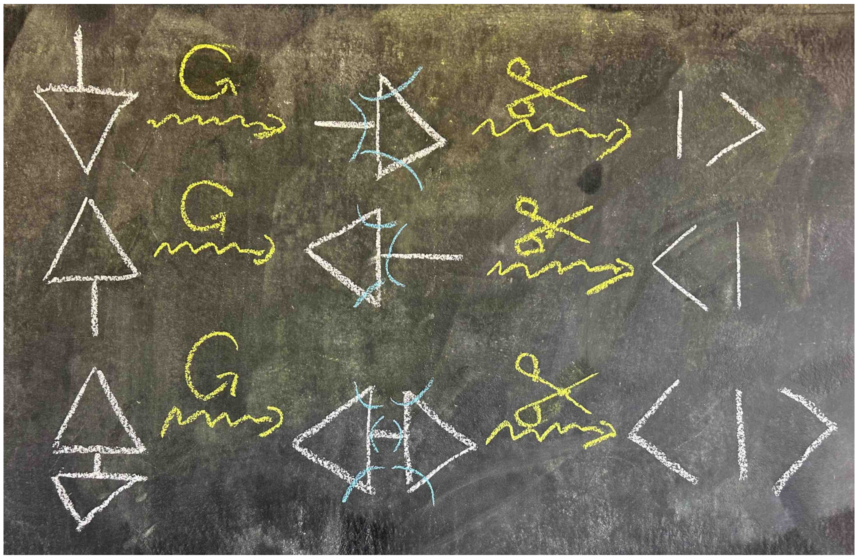


where these boxes will have some input and output wires. Here the wires represent quantum systems, or classical systems — cf. a place were quantum particles and classical data live/travel. The boxes transform those particles and the data. So for example, the boxes could be a unitary gate, or a quantum measurement, or encoding classical data as a quantum state etc. We take the convention that time flows upwards, that is, inputs are at the bottom and outputs are at the top.

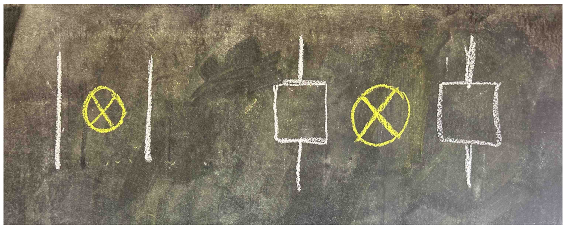
There are special kinds of boxes, called states and effects (or tests), and these respectively correspond to boxes without inputs and boxes without outputs:



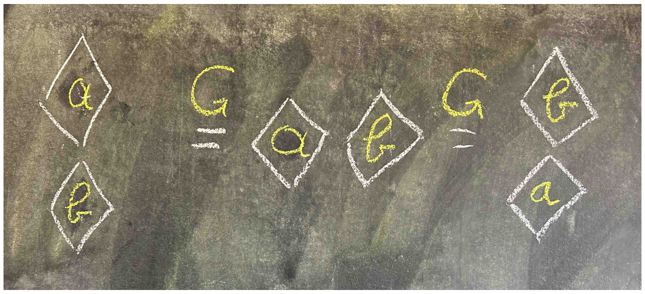
We depicted them as triangles, since then they resemble Dirac notation as follows:



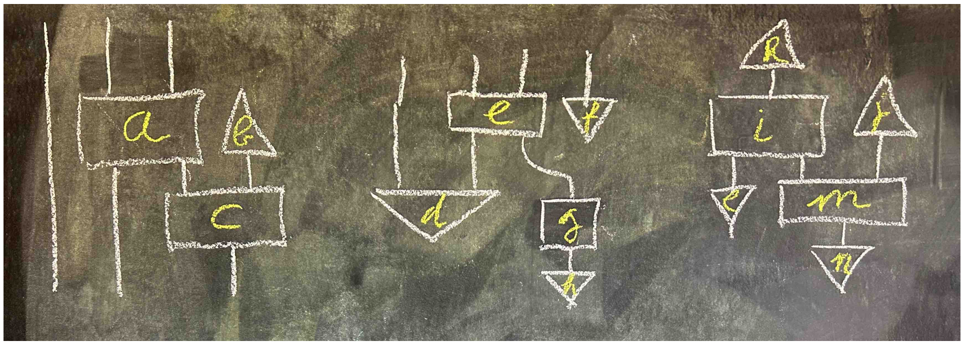
We can indeed think of these process theories as a 2D extension of Dirac notation, where putting two wires and/or boxes side-by-side corresponds to the tensor product:



We drew tensor symbols here, but in pictures, these tensor symbols are just empty space. Note also that in process theories numbers correspond to boxes without any input or output, denoted as a diamond, again because of Dirac notation. Since there are no external wires constraining them, they can freely move about:



We also saw how we can wire boxes together in order to form diagrams. Here are examples of a proper process, a state, and a number as diagrams:

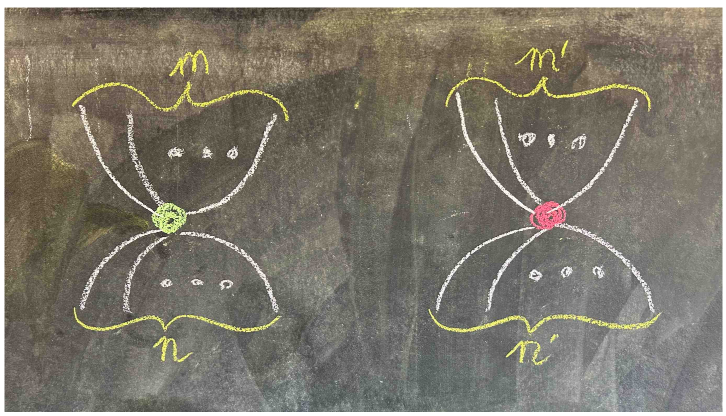


3 ZX-notation for linear maps

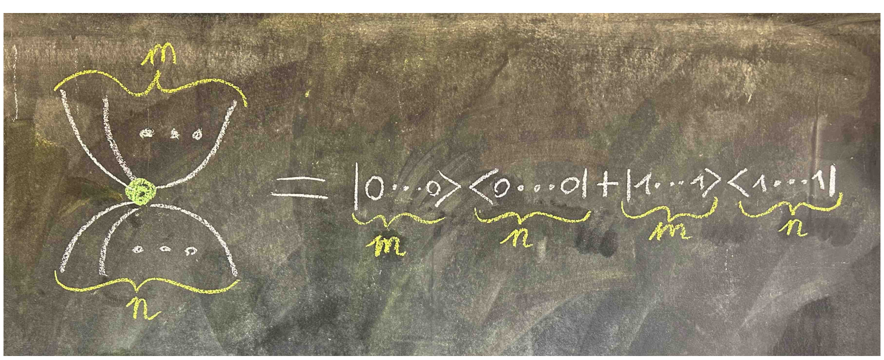
Above the boxes are just abstract boxes, or, black boxes if you wish. We will now fill them in with certain linear maps, which we call ZX-notation.

Plain spiders

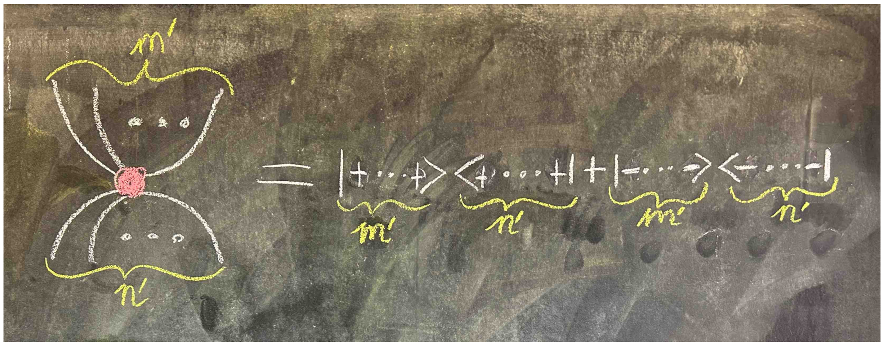
ZX-notation has mainly two symbols, called Z-spiders (green) and X-spiders (red):



They represent the following linear maps respectively:



and:



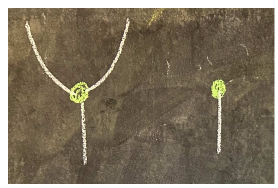
So they only differ in the basis in which they are expressed, respectively:

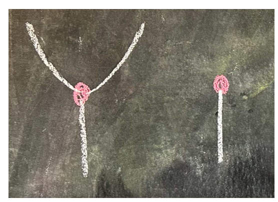
$$\text{Z-basis} = \{|0\rangle, |1\rangle\}$$

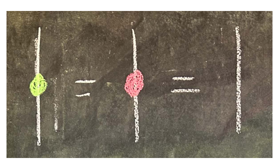
and:

$$\text{X-basis} = \left\{ |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \ , \ |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \right\}$$

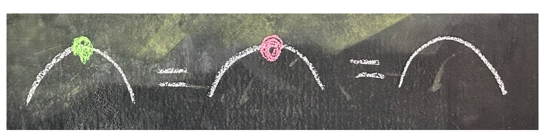
Here are some important examples of such spiders:

$\{|0\rangle, |1\rangle\}$-copy/delete = 

$\{|+\rangle, |-\rangle\}$-copy/delete = 

identity = 

Bell-state or cup = 

Bell-effect or cap = 

GHZ-state = 

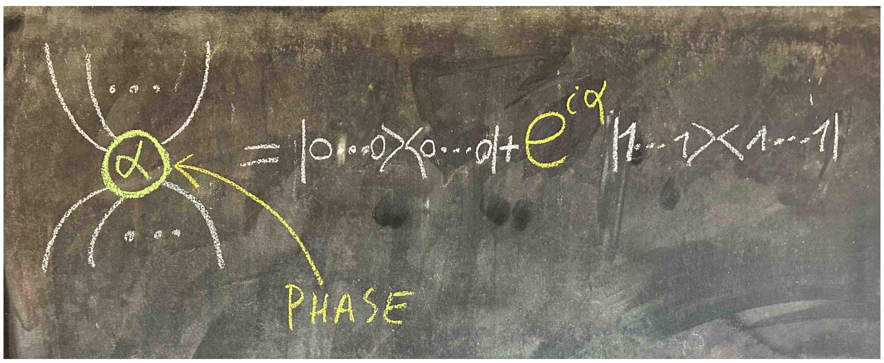
Note first that the states here are not normalised, which has calculational advantages.

Exercise. Show that the cup-spiders and cap-spiders are indeed the same for both colours. Convince yourself that except for these, the plain wire, and the spider with no legs, no other spiders of different colours coincide.

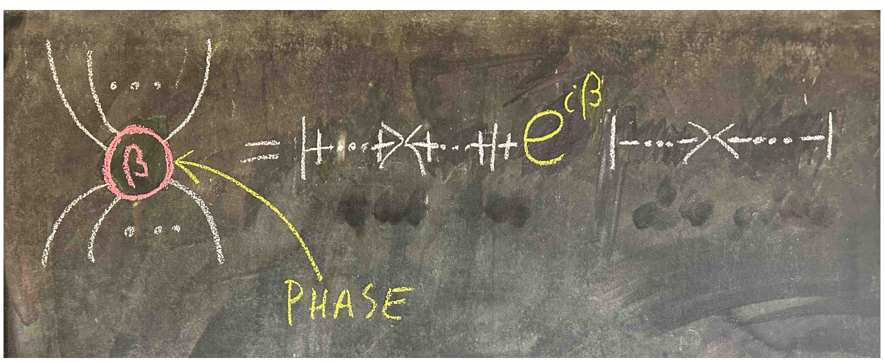
Exercise. Show that the spider with no legs is the number 2, and that it is equal to a circle made up of a cup and a cap.

Spiders with phases

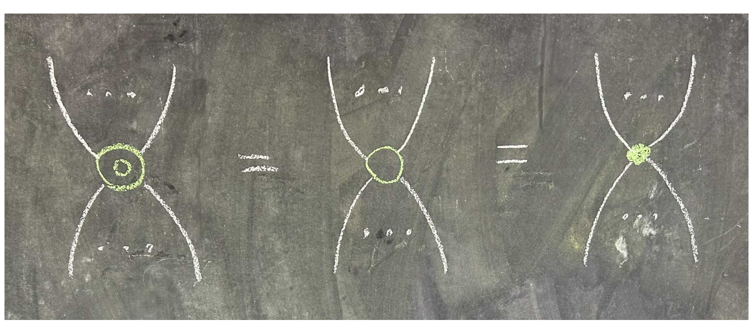
We will now also allow these spiders to carry phases, that is, a number in $[0, 2\pi]$, which modify the linear maps as follows:



and:



We typically don't write the 0-phase:

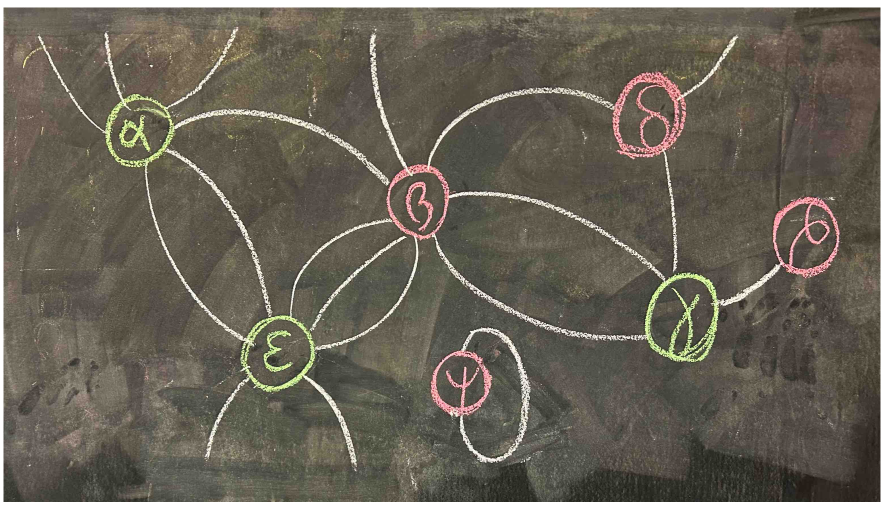


Some examples here are:

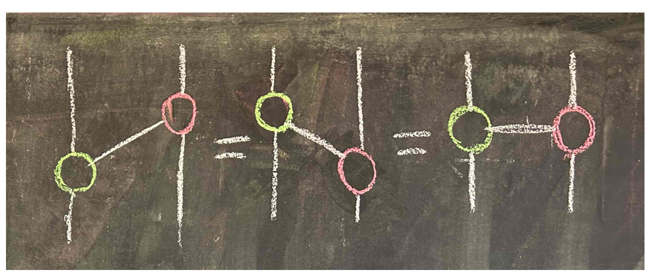
phase-gate = 

NOT-gate = 

A ZX-diagram is a ‘web’ made up of Z-spiders and X-spiders:

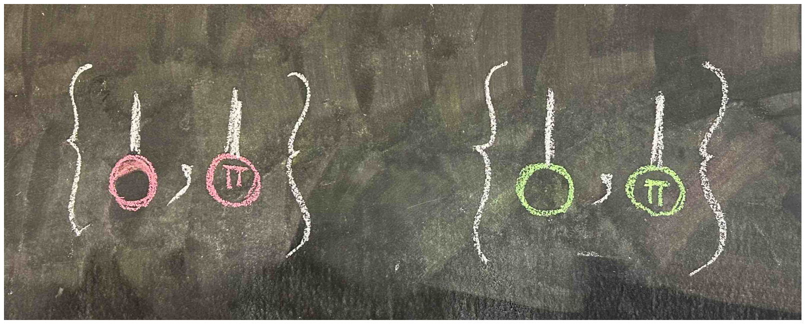


When we combine spiders of two colours the most important example is:

CNOT-gate = 

Exercise. Show that the two ways of defining the CNOT-gate are indeed the same, hence justifying the horizontal wire.

Also, up to normalisation, we can write all basis states as spiders:



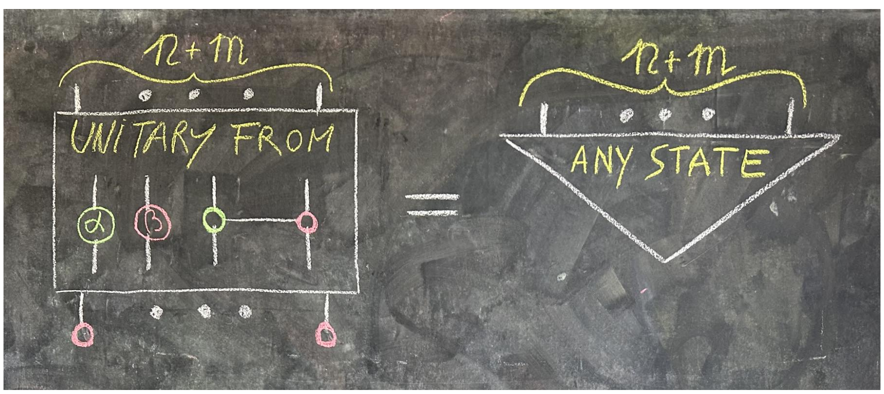
Note here that while the green spiders are built in terms of the Z-basis, its basis vectors are expressed in terms of the X-spiders, and vice versa.

Exercise. Show that this is indeed the case.

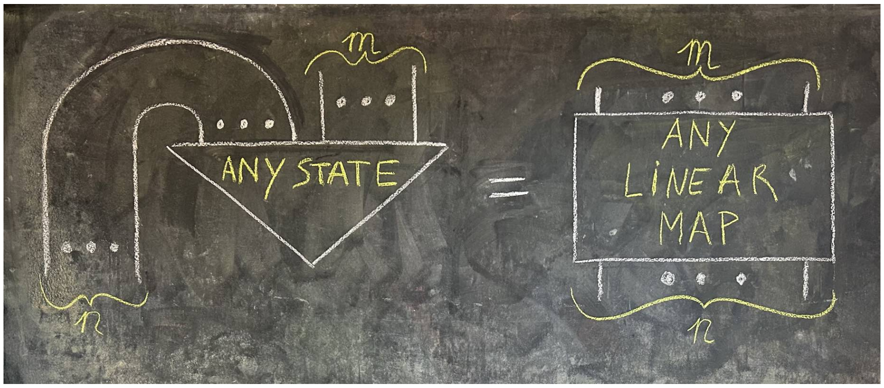
4 Universality of ZX-notation

Theorem. Any linear map between qubits can be written in terms of spiders.

In other words, ZX-notation is universal [5, 9, 14]. We can prove this as follows. It is well-known that any unitary map between qubits can be written in terms of CNOT-gates, Z-phase gates and X-phase gates, and each of these can be written in terms of ZX-spiders as we have seen above. This means that we can take a state of $n+m$ qubits expressible in terms of spiders, e.g. $|0\ldots0\rangle$, and produce any other $n+m$ qubit state by means of an appropriate unitary:



Then we can use cups to turn that arbitrary $n+m$ qubit state into a n qubit to m qubit linear map:

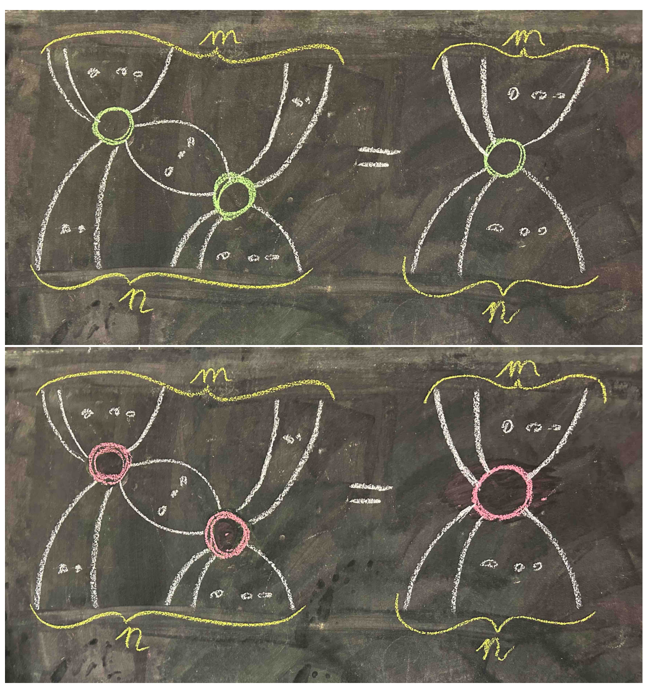


By the isomorphism between $n+m$ qubit states and n qubit to m qubit linear maps, we can obtain any n qubit to m qubit linear map by using caps. A more detailed diagrammatic account on this isomorphism can be found in [13, §3.1] and [14, §4.1.2].

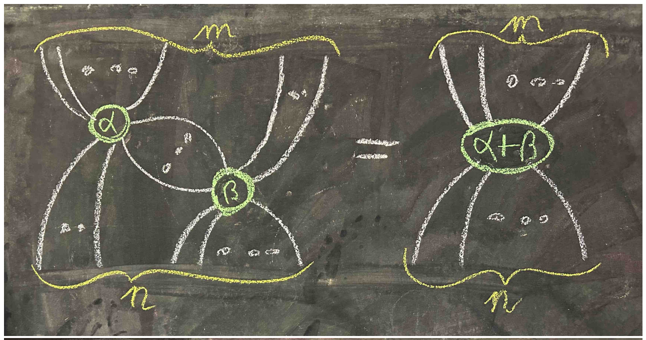
5 Rules of ZX-calculus

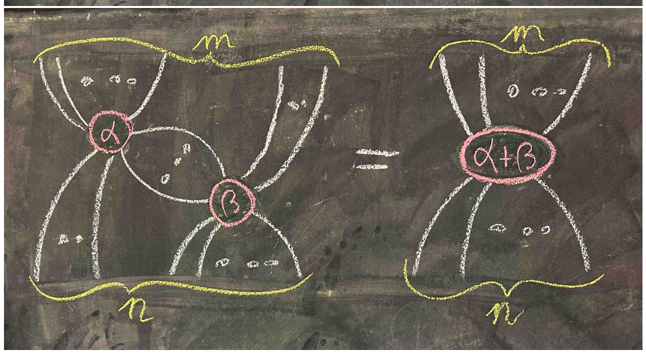
Spider-fusion

What makes ZX-calculus truly useful are the simple rules that allow us to transform ZX-diagrams. For spiders of the same colour without phases we have the following rule:



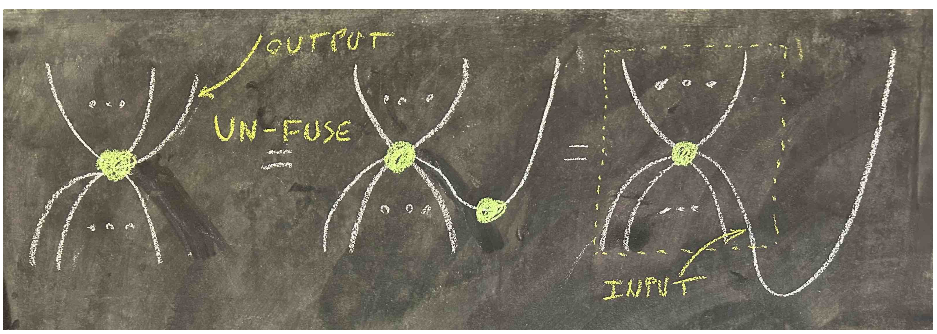
which tells us that spiders of the same colour fuse together. This also means that any connected diagram of spiders of the same colour only depends on its number of inputs and outputs, since each such web is itself a spider. If we have spiders with phases than these phases add up when fusing:





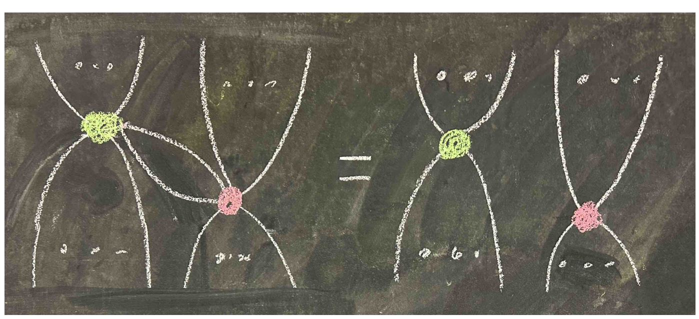
Exercise. Prove the fusion rule for spiders with phases.

Remark. From spider fusion it follows that inside a ZX-diagram one can always ignore what is an input and what is an output of a spider, as we can freely turn one into the other by (un-)fusing, using either a cup or cap as follows:

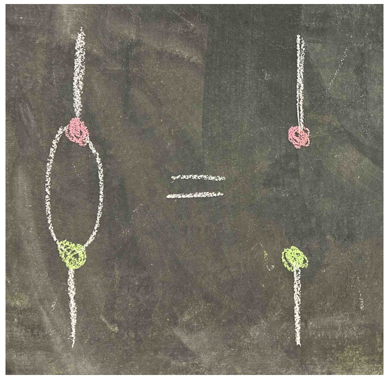


Hopf-rule (or leg-chopping) and bialgebra-rule (or square-popping)

When spiders of different colours share two legs, then these vanish:



which, thanks to spider-fusion, can be simplified as follows:

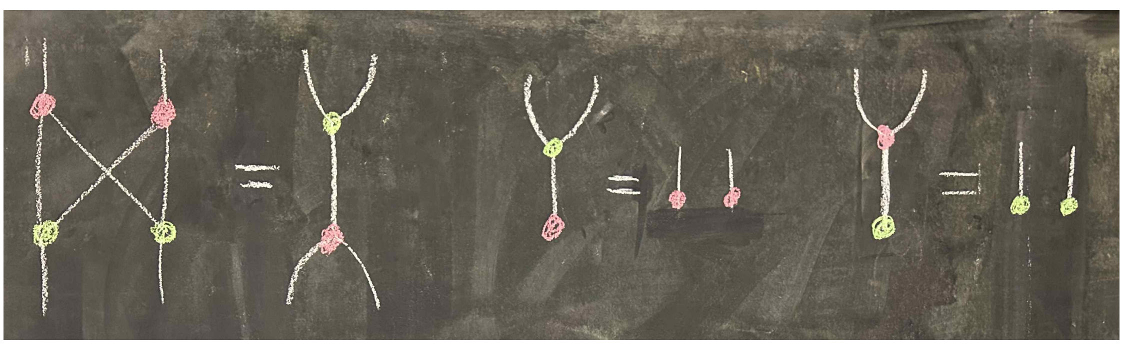


For historical reasons we call this the Hopf-rule. In [11] we call it leg-chopping.

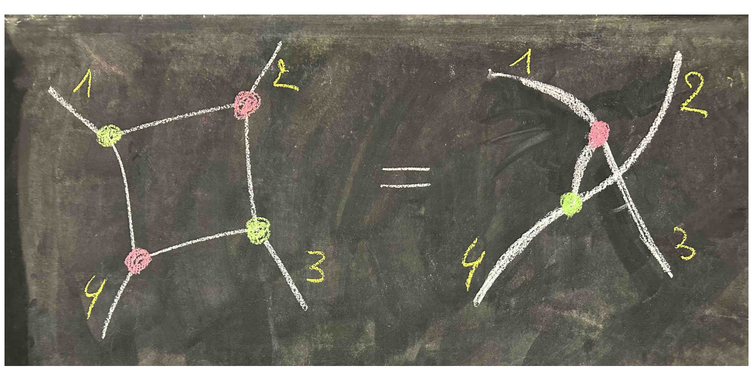
Remark. This equation is only true up to a number, but for the sake of the argument we won't go into this issue here. In fact, as long as you are not computing a probability, then you can typically always figure out at the end of the calculation what the normalising number should be [14, §3.4.3].

Exercise. Prove the Hopf-rule.

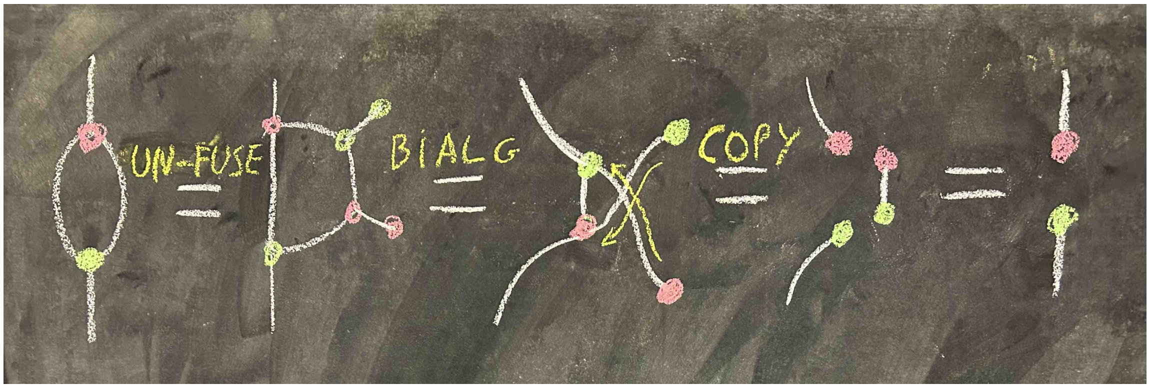
There is a rule which is stronger than the Hopf-rule and consists of three equations:



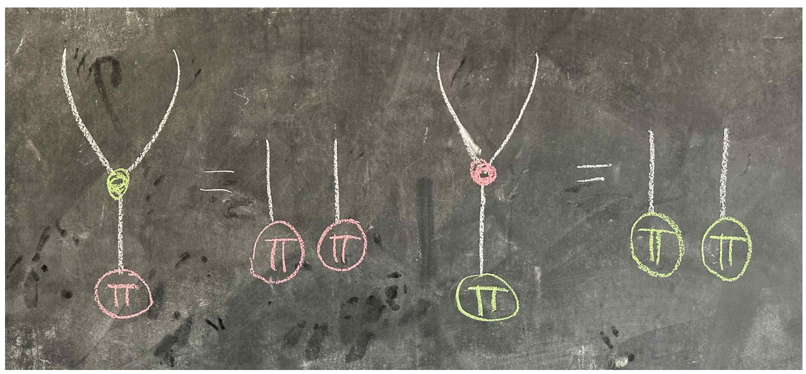
For historical reasons we call the first of these the bialgebra-rule. The two two copy-rules say that "green copies red", and that "red copies green". In practice, the way the 1st one of these is more often used is in the following form, which gets rid of squares, while leg-chopping only gets rid of 2-cycles:



Therefore in [11] it is called square-popping. One can derive the Hopf-rule from the bialgebra-rule and the two copy-rules:



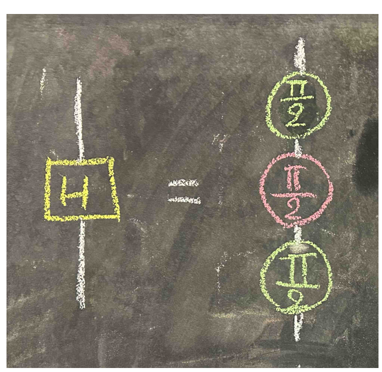
Using some group theory, which we won't do here and you can find in [14, §9.3.4], one can also show that the other basis states are also copied by the opposite colour:



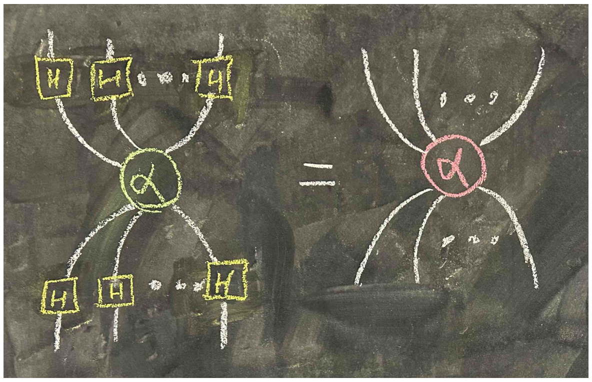
The rules listed above will get you a very long way already, including some cutting-edge new quantum computing techniques, as we will see further below. But we will first present a larger set of rules about which we can prove something very important.

Colour-change

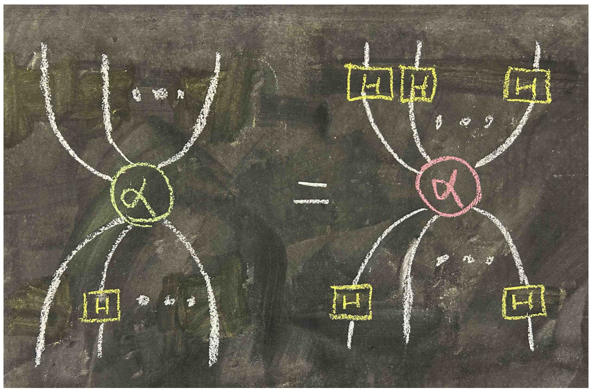
It is useful to have a special notation for the Hadamard-gate which in terms of Euler angles can be written as follows in ZX-notation:



One extra rule then is called colour-change:



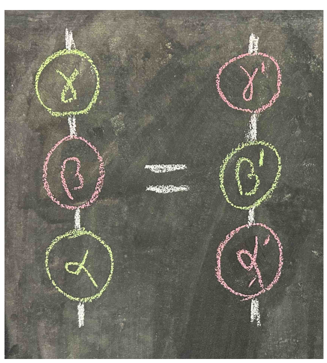
Equivalently, an H-box can be copied through a spider provided its colour changes:



Exercise. Prove the colour-change rule.

Phase-colour swap

A final rule takes the shape:



and we call it phase-colour swap [16, 33], but we won't specify here the relation between the angles as we won't use it. In fact, it is not very often used in practical applications.

6 Completeness of ZX-calculus

Theorem. Any equation that holds for linear maps between qubits and compositions thereof can be derived in ZX-calculus.

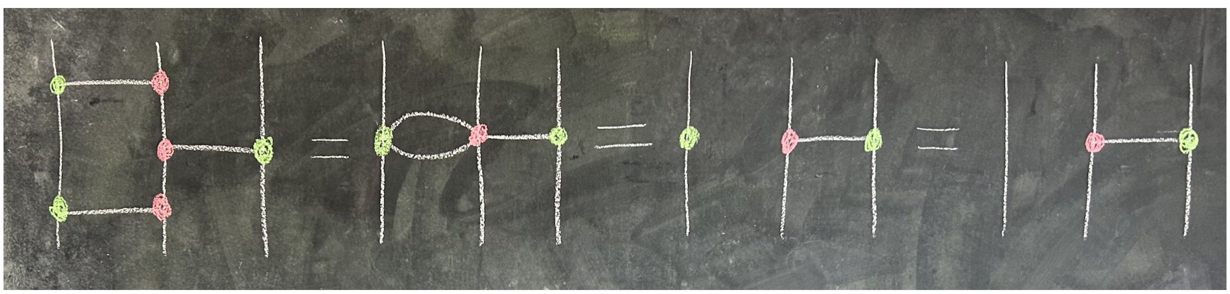
In other words, ZX-calculus is complete for linear maps. The proof here is hard, and it took a string of papers to get to this point [1, 2, 23, 25, 24, 33].

7 Doing stuff with ZX-calculus part I

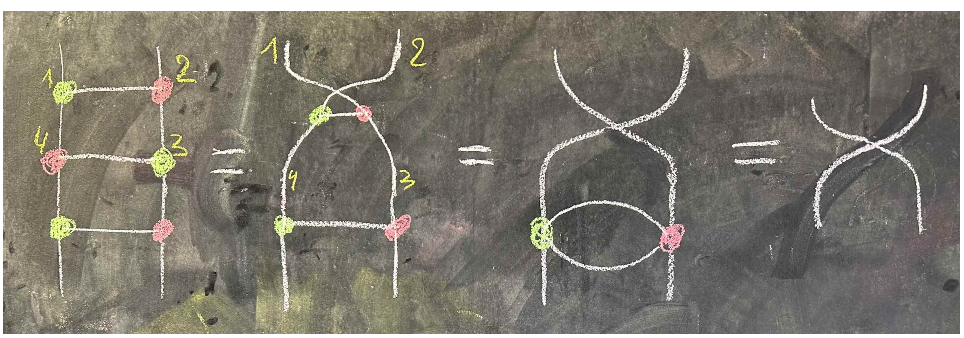
We will now see ZX-calculus in action. The point of the protocols described below is not to give a detailed discussion of them, but to illustrate how ZX-calculus helps to derive or verify them. We provide links to the original papers so you can compare descriptions.

7.1 Stuff with quantum circuits

We can now use ZX-calculus to simplify this quantum circuit:



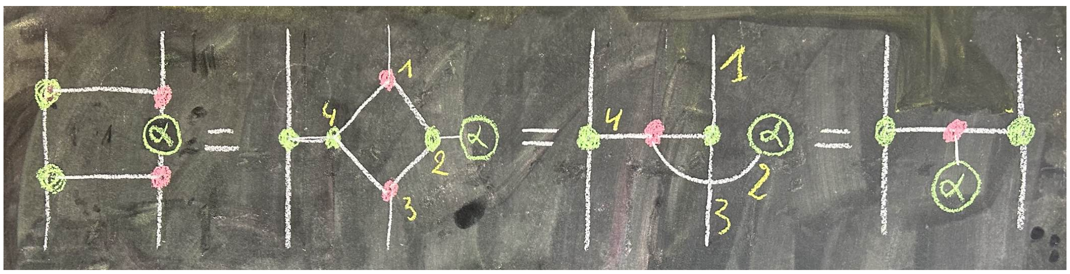
Or this quantum circuit:



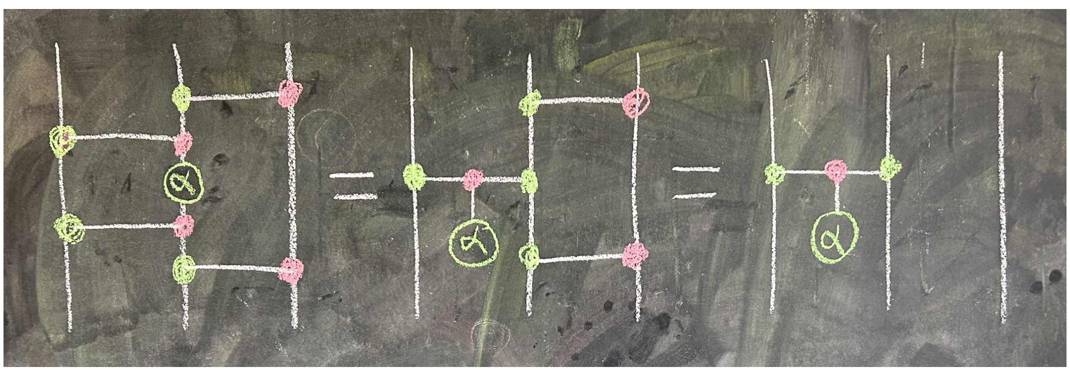
Or we can use it to show that a CNOT-gate indeed does what a CNOT-gate should do:

and:

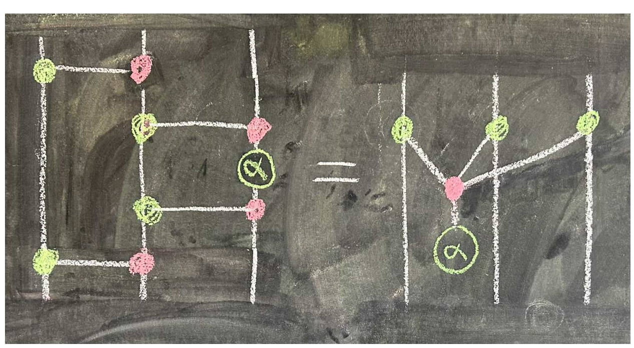
These were very simple examples, and now we will show a generally applicable technique that is used in quantum circuit optimisation, e.g. within the tket-compiler [17]. The following circuit fragment has a 'hidden' square and we can simplify it:



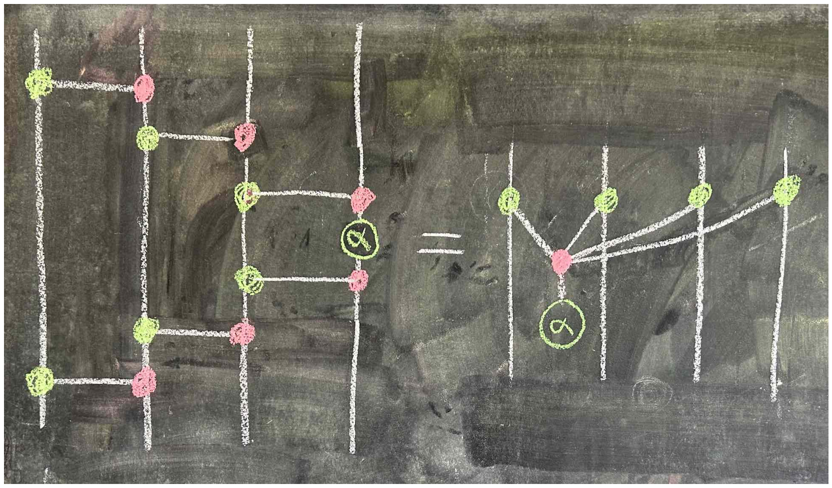
We now only have green spiders on the vertical wires, which allows to, for example, simplify the following circuit:



Similar tricks moreover scale up, cf.:



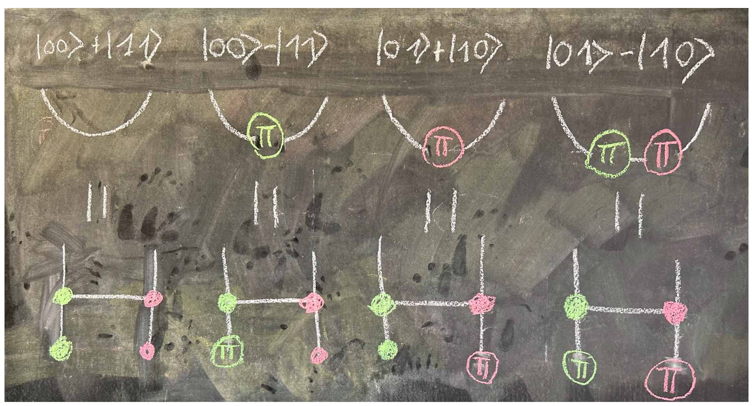
and:



and so on.

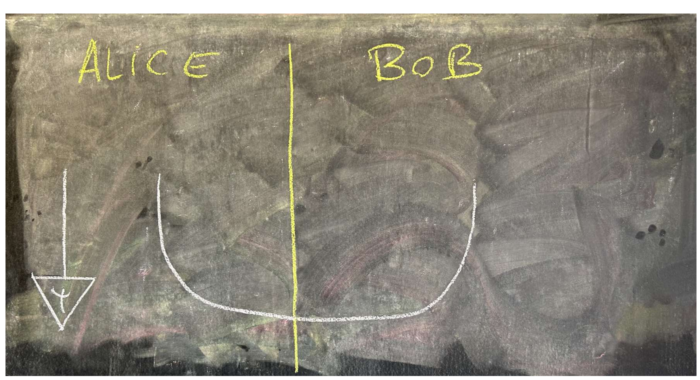
7.2 Quantum teleportation

We can represent the Bell-basis as follows:

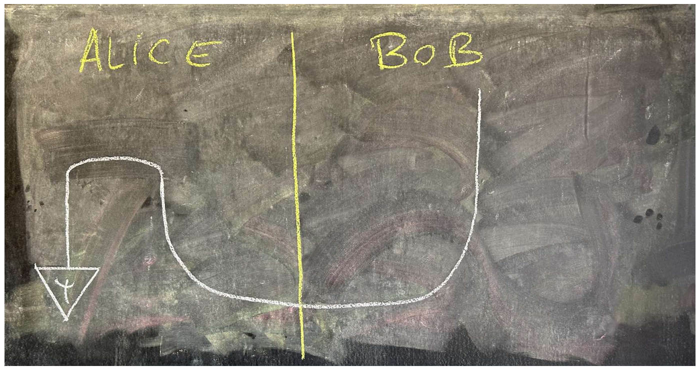


Exercise. Verify that this is true.

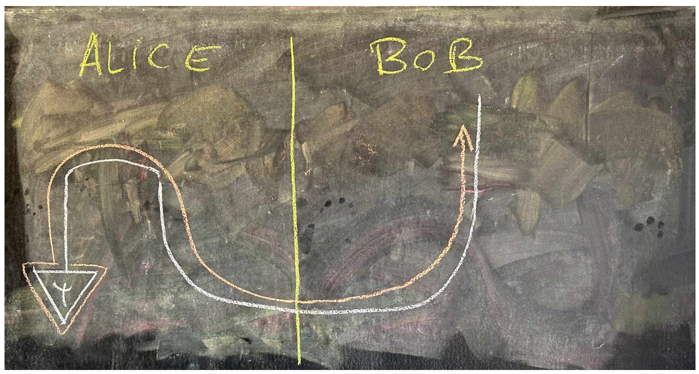
Now, suppose Alice and Bob share a Bell-state, Alice has a qubit in state $|\psi\rangle$, and Bob needs that state. How can that state be passed on without effectively sending the qubit:



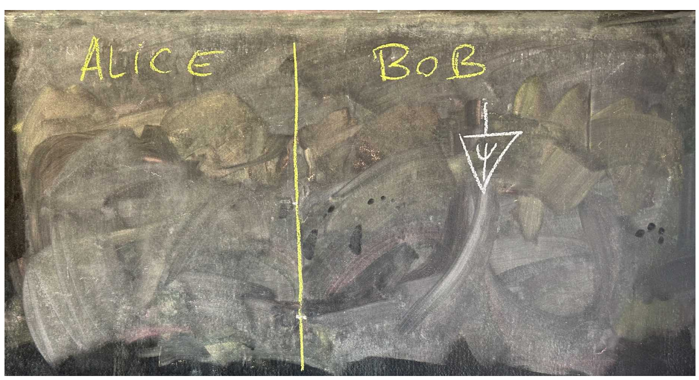
An obvious solution now presents itself: Alice applies a Bell-effect:



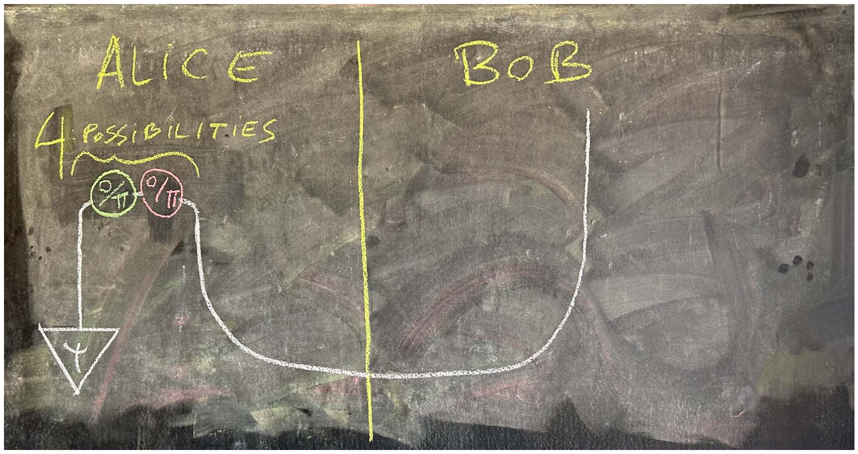
We now indeed can slide the state along the wire:



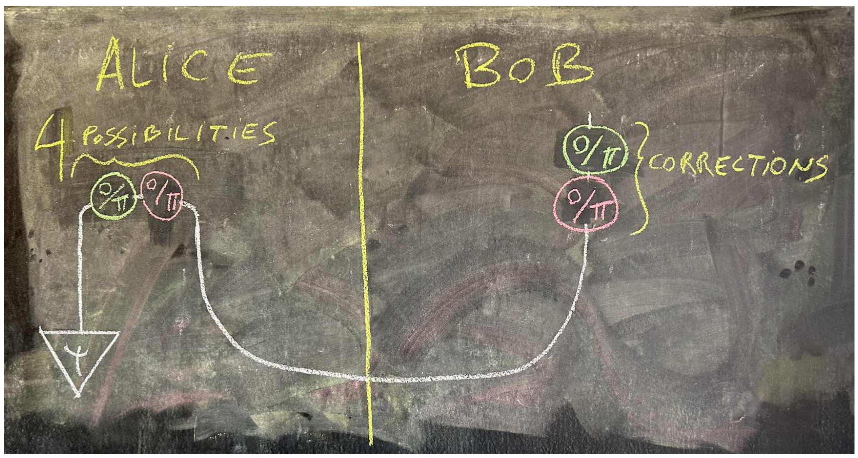
so that it ends up with Bob:



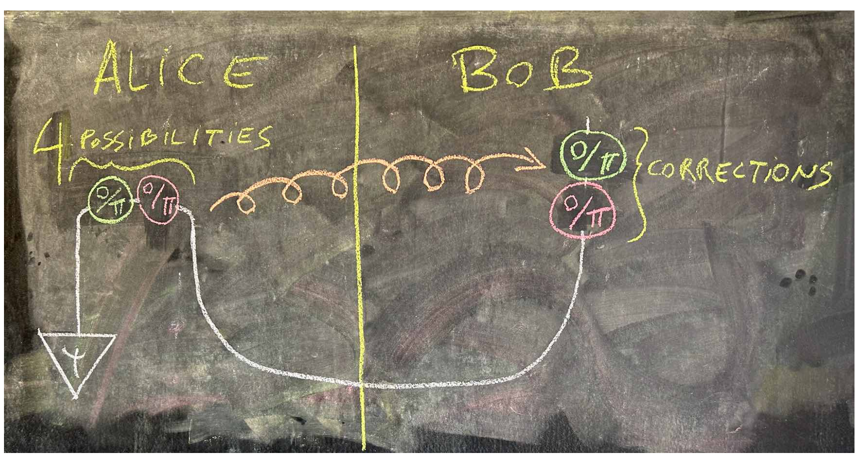
But we can only achieve a Bell-effect as part of a measurement against the Bell basis, which would yield one of the four Bell-effects:



Fortunately, we can undo the corresponding unitaries at Bob's end:



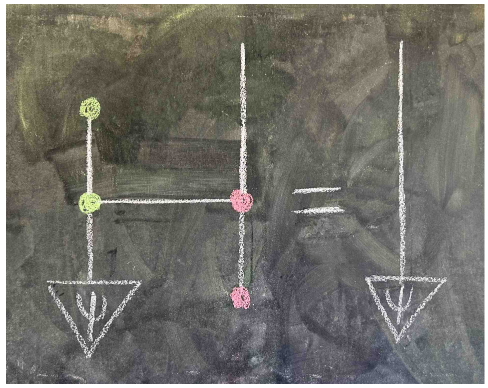
which requires communicating two bits from Alice to Bob:



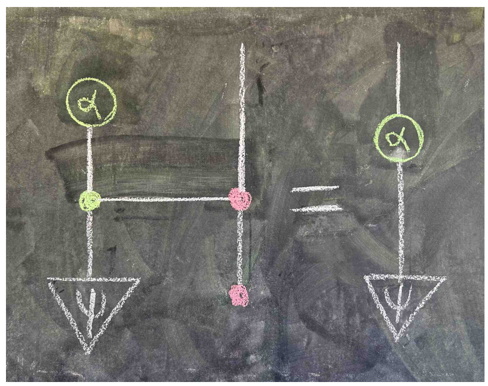
The original paper on quantum teleportation is [4].

7.3 Measurement-based quantum computing

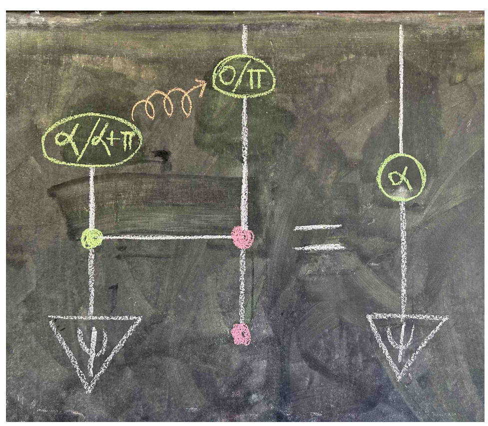
We can also pass a state as follows, which now only involves two qubits:



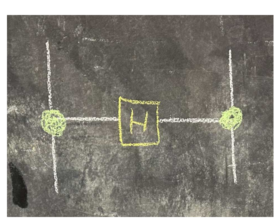
If instead of an X-measurement, we use a measurement corresponding to a basis with a phase, then that phase is applied to the state as a phase gate:



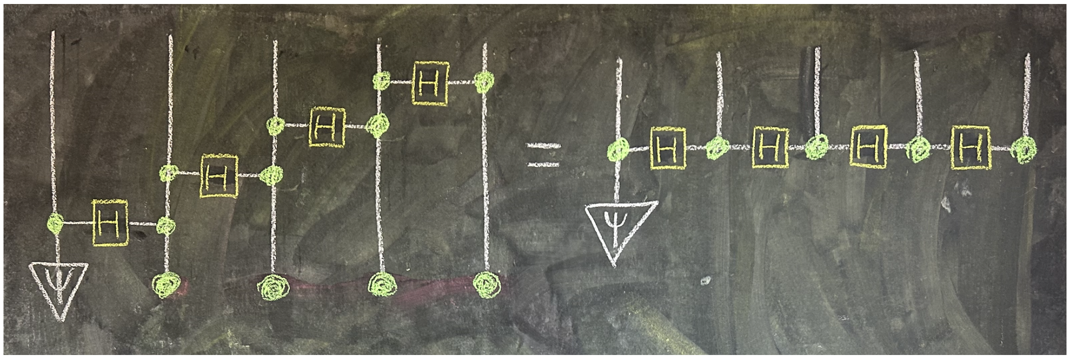
Again accounting for two possible measurement outcomes, we can do:



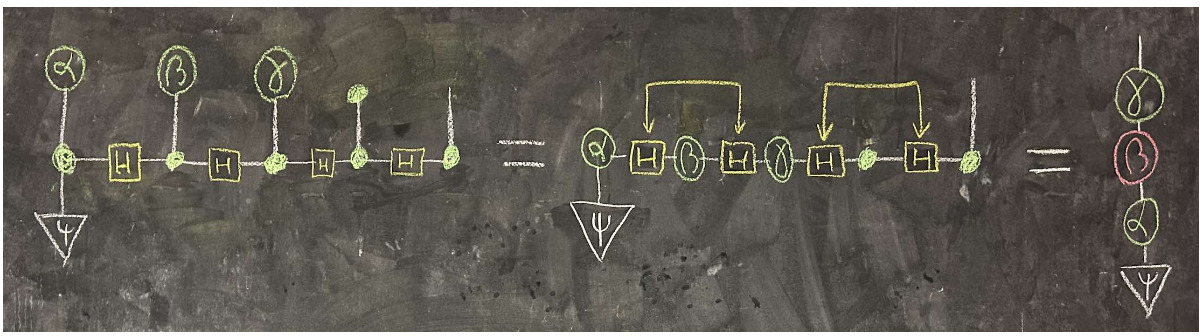
and the gate is now applied for every measurement outcome. This manner of applying a phase gate is called measurement-based quantum computing. Using the following gate:

CZ-gate = 

we can make the following state:



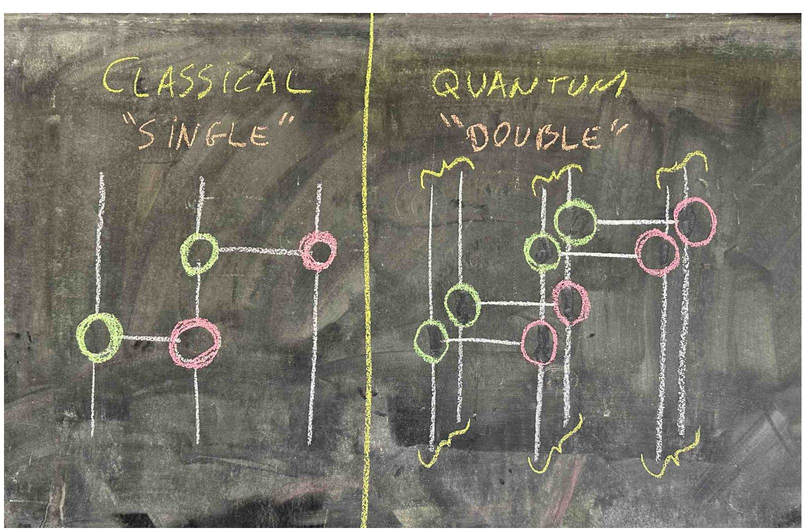
Now, using measurement-based quantum computing we obtain:



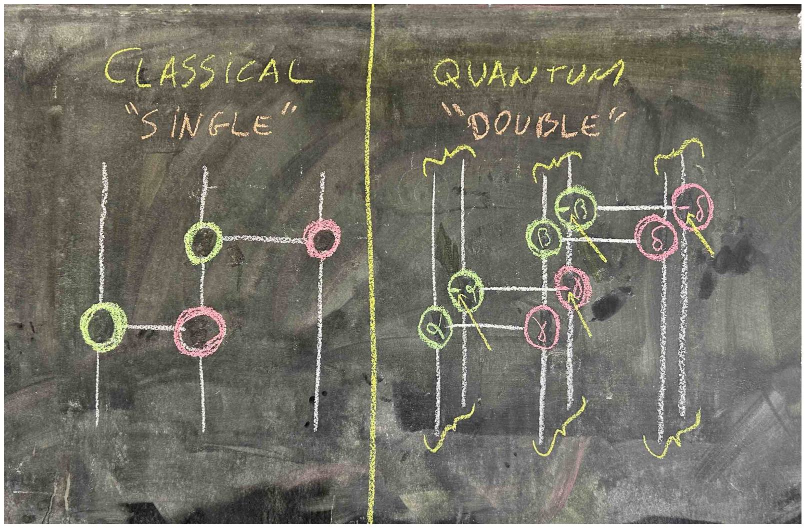
that is, we have applied an arbitrary unitary in terms of its Euler angles. The two MBQC protocols presented here were first introduced in [30] and [31] respectively.

8 Classical versus quantum in diagrams

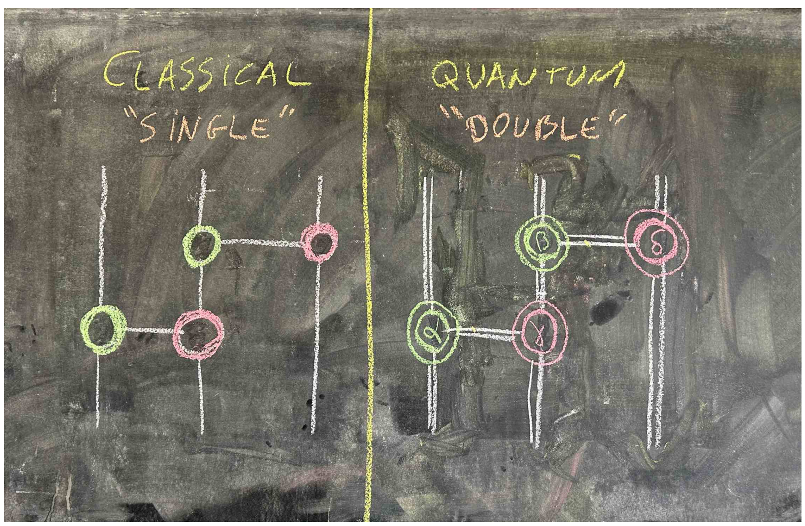
In order to distinguish between classical and quantum diagrams, we draw classical diagrams as single diagrams, and quantum diagrams as doubled diagrams:



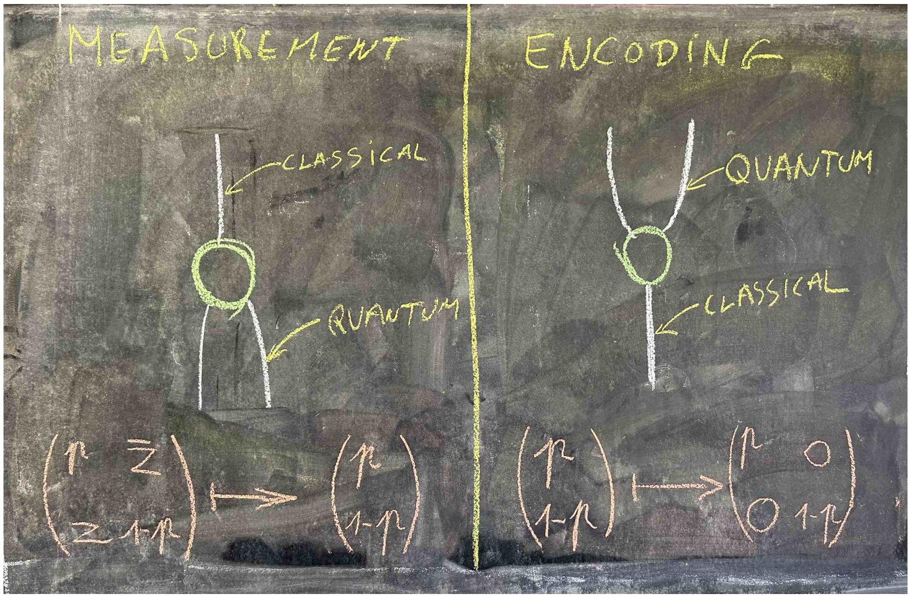
If there are phases, then the double needs a minus:



To make drawing easy we just double wires and spider-nodes:



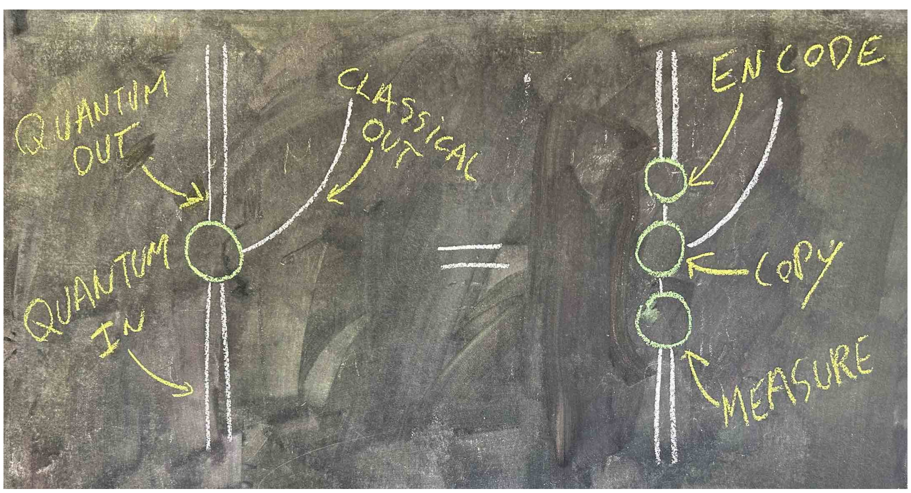
A measurement goes from quantum to classical and we can also encode classical data as a quantum state as follows:



As indicated, a measurement takes in a density matrix and returns a classical probability distribution on the possible outcomes, and encoding takes in a classical probability distributions and returns a density matrix with that probability distribution on the diagonal.

Exercise. Show that this indeed the case. To do so, one should think of the density matrix as a two qubit state via the isomorphism between qubit to qubit linear maps and two qubit states.

Non-demolition measurements are depicted as follows:



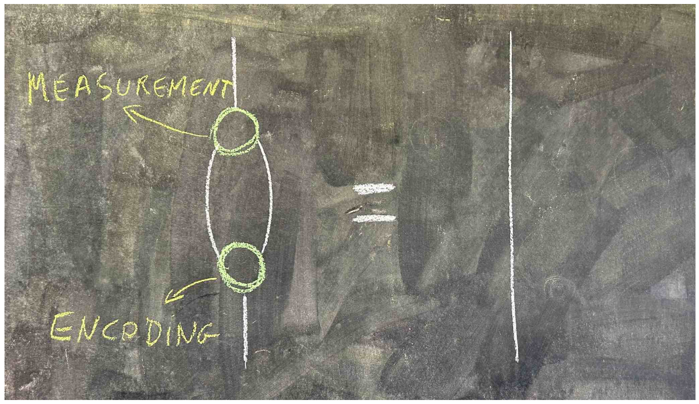
where the decomposition on the right clearly shows how collapse is implicit here.

9 Doing stuff with ZX-calculus part II

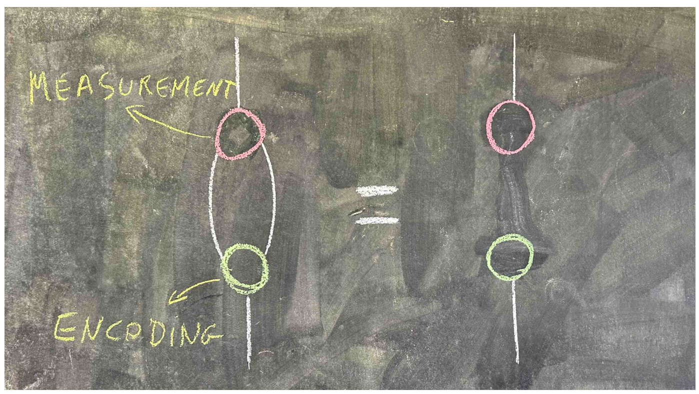
We will now see some more ZX-calculus in action.

9.1 Composing quantum measurements

Measurement after encoding in the same colour by spider-fusion results in doing nothing:

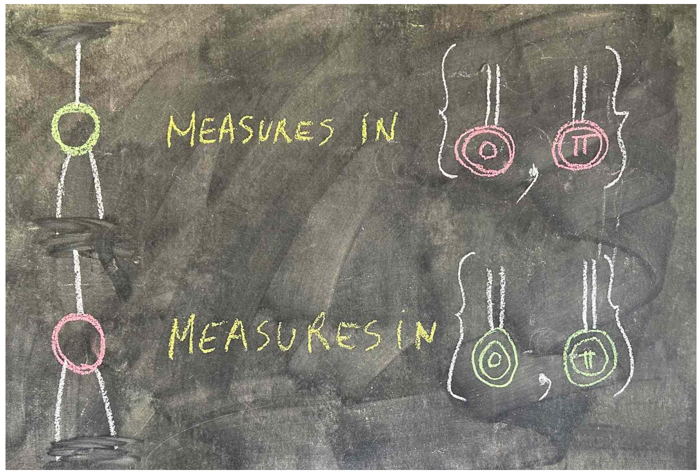


while by the Hopf-rule for different colours we get:

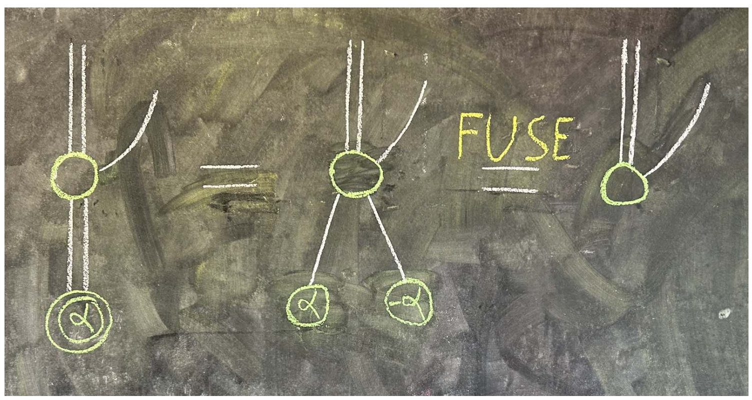


that is, nothing gets through. In fact, the classical states:

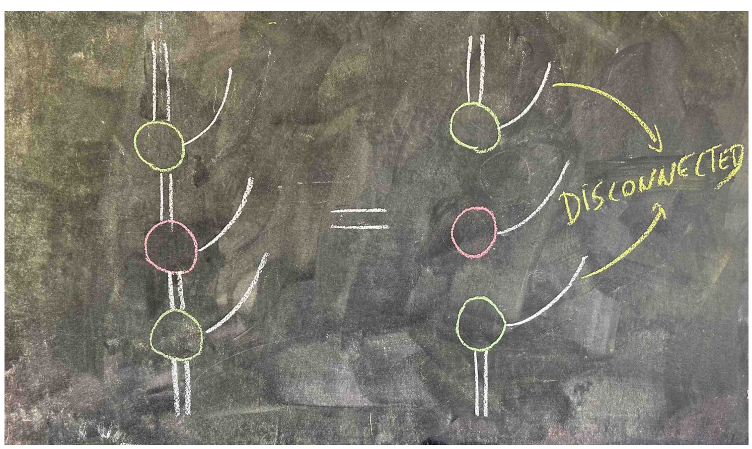
stand for the uniform probability distribution, for classical data obtained in a Z-measurement and an X-measurement respectively. In our ZX-representation it is indeed the case that classical data is encoded in the basis that it is measured in [14, §9.2.4], recalling that:



When we measure a phase:



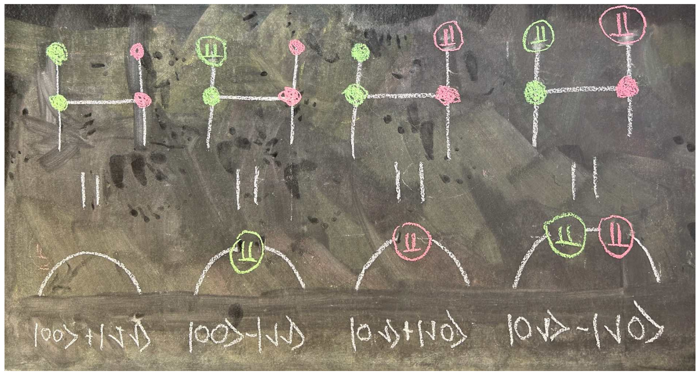
it vanishes, and if we measure Z, then X, and then Z again:



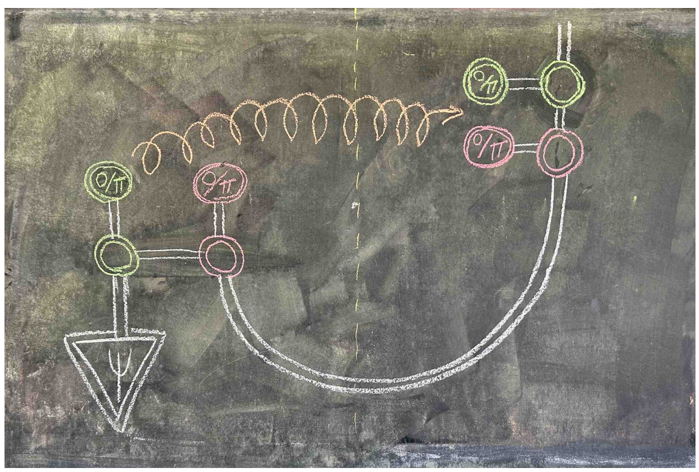
so the two outcomes of the Z-measurements are unrelated. As shown in [15], this simple results is directly underpinning the security of quantum key distribution in the BB84 form [3], and by using cups we obtain the Ekert 91 form [21].

9.2 Classical communication

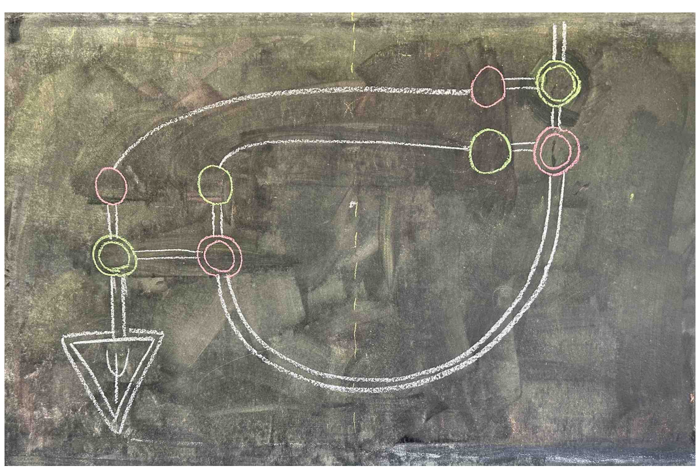
We now revisit quantum teleportation again in order to make the communication between Alice and Bob part of the diagram. Flipping what we saw earlier upside down, we can represent the Bell-measurement as a CNOT, a Z-measurement and an X-measurement:



so we obtain:

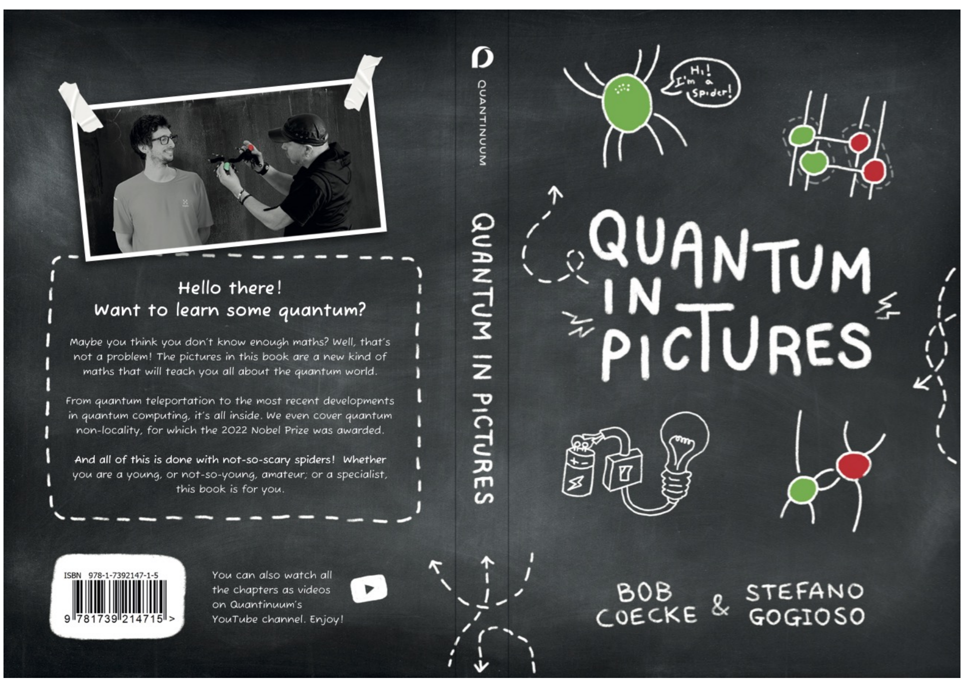


and now using spiders as measurements and encoding we get:



where the colour change is due to the fact that green/red measures in the red/green basis. We now know what the dodo on the cover of [14] was all the time thinking of:

If you can't get enough of ZX-calculus, then this 'dodo-book' is also a place to continue reading. In the case that its 900 pages scare you – do note that we teach it in only 20 lectures at Oxford University – you may want to start with Quantum in Pictures [11]:



Both [14] and [11] contain a purely diagrammatic proof of quantum non-locality, and lot's of other stuff not presented here. In fact, QiP tells the pictorial ZX-story without any reference to Hilbert space nor any other mathematics or physics prerequisites. The very fact that this is even possible is an important statement in its own right, both from a scientific and educational perspective. A somewhat similar to this one but older tutorial is [10]. An overview of some more recent applications of ZX-calculus and also some more philosophy can be found in [12]. Professionals may also want to go for the more advanced [32]. All the ZX-publications can be found at:

https://zxcalculus.com/publications.html

as well as a link for the ZX-calculus Discord page.

10 Acknowledgements

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